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# THE CONSTRAINT METHOD - A NEW FINITE ELEMENT TECHNIQUE

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## SUMMARY

A new approach to the finite element method which utilizes families of conforming finite elements based on complete polynomials is presented here. Finite element approximations based on this method converge with respect to progressively reduced element sizes as well as with respect to progressively increasing orders of approximation. Numerical results of static and dynamic applications of plates are presented to demonstrate the efficiency of the method. Comparisons are made with plate elements in NASTRAN and the high-precision plate element developed by Cowper and his co-workers. Some considerations are given to implementation of the constraint method into general purpose computer programs such as NASTRAN.

## INTRODUCTION

With the availability of general purpose computer programs, such as NASTRAN, at reasonable cost, utilization of the finite element approximations is common practice. In the conventional finite element method, a continuous structure is idealized by discrete structural elements which are joined together at nodes. Structural characteristics are expressed in terms of nodal variables. Improvement of accuracy is generally made with respect to progressively reduced element sizes. If certain conditions are met, then the finite element approximation will converge to the true solution when the element sizes are reduced (Ref. 1).

Unfortunately, reliable and practical error estimation techniques are not yet available. In important analytical computations it is usually necessary to complete two or more calculations of the same problem in order to establish the validity of the finite element model itself. The calculations usually employ progressively refined finite element nets, although upper and lower bound estimates have been proposed also (Ref. 2). This is a tedious and costly process involving a considerable amount of duplicated effort.

Much interest has been shown in the development of high-precision finite elements so that better accuracy could be obtained with fewer elements. Incorporation of such elements into the NASTRAN program was reported to be under development (Refs. 3 and 4).

Computational experiments as well as theoretical considerations have shown that, in terms of the number of variables needed to carry an analysis to a specified level of precision, the high-order or high-precision finite elements are more efficient than the low-order ones. This is particularly true in vibration and buckling analyses where eigenvalue problems must be solved in terms of the problem variables. An additional fact in favor of using fewer high-precision finite elements is that the number of necessary man-computer interface operations and the volume of data processing services are roughly proportional to the number of finite elements employed.

In view of these findings it is logical to explore potential benefits to be gained from the convergence process based on progressively increasing orders of polynomial approximation. In this convergence process, the finite element net is held constant and the order of polynomial approximating functions is varied. Existing error bounds such as that proposed by Fried (Ref. 5) indicate that the convergence rate will be exponential in this case, whereas the convergence rate is geometrical when the finite element sizes are reduced. It is noted that this error bound is valid only when the exact solution is sufficiently smooth and free from singularities.

While there are many competing formats for stating finite element approximation problems, it was found that it is convenient to state the general problem as a quadratic programming problem. In this formulation, which will be referred to as the constraint method in the following discussions, the functional to be approximated (usually the potential energy expression) is written as a quadratic expression of the unknown coefficients of the polynomial approximating functions. The interelement continuity conditions and principal boundary conditions are stated as linear equality constraints. An advantage of this formulation is that all matrices that are necessary to define the numerical problem can be generated automatically for arbitrary orders of approximation. The finite elements so constructed will exhibit convergence with respect to reduced element sizes as well as with respect to increasing orders of polynomial approximation. Because of the latter type of convergence, it is unnecessary to reconstruct the finite element model when higher accuracy is sought. Additional advantages of this formulation are: (a) Since the unknowns are the scalar coefficients of the approximating polynomial sequences, it is not necessary to transform the variables and stiffness matrices from one coordinate system to another. (b) All finite elements can be made mutually compatible by specifying the appropriate connectivity through the constraint equations. This is a very important feature of the new method because it permits consideration of structural stiffeners with greater ease than the standard finite element methods. (c) The new method will yield the exact solution when the exact solution is a polynomial with a degree less than or equal to the degree of the approximating polynomials, regardless of the number or orientation of the

finite element employed. (d) The accuracy of solution and the computational efficiency are not sensitive to input numbering schemes (therefor, the method provides flexibility to the users in generating the structural models). (e) The number of elements depends solely on the geometrical configuration of the structure to be analyzed, not on the desired degree of precision as in conventional analysis. Thus only the minimum number of elements sufficient for idealizing the structure needed to be defined. Other factors, such as existence of point loads and/or discrete supports, do not preclude the use of large elements.

In the following, a solution technique utilizing the essential features of this formulation is discussed and applications of the constraint method are illustrated with numerical examples for static and dynamic analysis of plates. Comparison is made with results obtained by plate elements in the NASTRAN program and the 18 degrees-of-freedom plate element presented by Cowper et al. (Ref. 6).

#### THE CONSTRAINT METHOD

In the constraint method, the finite element approximation is treated as a direct energy minimization problem in which the minimum potential energy is sought subject to certain linear constraints. As in the conventional finite element method, the structure is idealized by discrete elements whose displacement characteristics are approximated by the polynomial functions defined over the element domains. Usually, the unknown variables are the coefficients in the assumed polynomials, although other variable definitions may be used also. The total potential energy is minimized with respect to these unknown coefficients subject to constraints which ensure satisfaction of both inter-element continuity and kinematic boundary conditions.

Detailed formulation of the constraint method has been presented elsewhere (Refs. 7 to 10). It will be outlined here as follows:

The total potential energy  $\pi$  is obtained by assembling the element potential energies  $\pi_k$ , expressed in terms of the coefficients of the approximating polynomials as

$$\pi = \sum_k \pi_k = 1/2 [a] [S] \{a\} - [Z] \{a\} \quad (1)$$

In equation (1),  $[a]$  is a row vector, containing the polynomial coefficients and  $\{a\}$  is transpose of  $[a]$ ;  $[S]$  is a symmetric, positive matrix containing a set of submatrices along its diagonal;  $[Z]$  is a row vector associated with applied loading. This equation is treated as a quadratic objective function which is to be minimized subject to the following linear constraints:

$$[P] \{a\} = \{R\} \quad (2)$$

where  $[P]$  and  $\{R\}$  define the interelement continuity and the external boundary conditions. For homogeneous boundary conditions  $\{R\}$  is null.

Several different algorithms can be used for solving the problem represented by equations (1) and (2). Most of these require separation of the independent variables from the dependent ones in the constraint equations. Then the problem can be reduced to solving a system of simultaneous linear algebraic equations as explained in Appendix A.

For structures subject to dynamic loading, inertia properties must be introduced in addition to the structural stiffnesses. In the case of free vibration, the equation of motion for the Kth element is expressed as

$$[S]_K \{a\}_K + [M]_K \{\ddot{a}\}_K = 0 \quad (3)$$

where  $[M]_K$  is the consistent mass matrix and  $\{\ddot{a}\}_K$  is the second derivative of  $\{a\}_K$  with respect to time. The unconstrained equations of motion for the entire system are obtained by summation and can be written as

$$[S] \{a\} + [M] \{\ddot{a}\} = 0 \quad (4)$$

After separating the independent and the dependent variables in the constraint set (see Appendix A), the unknown variables  $\{a\}$  can be expressed in terms of the free variables  $\{a_c\}$ , and the constrained equations of motion become

$$[H^T] [S] [H] \{a_c\} + [H^T] [M] [H] \{\ddot{a}_c\} = - [H^T] [S] \{h\} \quad (5)$$

The matrix  $[H]$  and the vectors  $\{a_c\}$  and  $\{h\}$  are defined in Appendix A.

For homogeneous boundary conditions,  $\{h\}$  vanishes and equation (5) becomes

$$[\bar{S}] \{a_c\} + [\bar{M}] \{\ddot{a}_c\} = 0 \quad (6)$$

where  $[\bar{S}]$  and  $[\bar{M}]$  are the constrained stiffness and mass matrices, respectively:

$$[\bar{S}] = [H^T] [S] [H] \quad (7)$$

$$[\bar{M}] = [H^T] [M] [H] \quad (8)$$

It is noted that the eigenvalue problem associated with equation (6) is relatively small since all dependent variables were eliminated.

An important feature of this formulation is that all finite element approximation problems can be fully defined by the matrices  $[S]$ ,  $\{Z\}$ ,  $[P]$ ,  $\{R\}$ , and  $[M]$  (for dynamic application) for arbitrary orders of approximation. These matrices can be generated automatically for any given problem.

### NUMERICAL EXAMPLES

The efficiency of the constraint method is illustrated with examples for static and dynamic analyses of structural plates. A comparison is made with results obtained using plate elements in the NASTRAN program and the high-precision plate element presented by Cowper et al. (Ref. 6). Emphasis is on the accuracy and convergence of the constraint method with respect to increasing orders of approximation and using a minimum number of elements. Additional numerical results for static analysis of plates and shells can be found in references 7 to 10.

#### Static Analysis

The first example problem for static application is the simply-supported equilateral triangular plate (Fig. 1(a)) under uniform pressure  $q$ . The exact solution of this problem is a 5th order polynomial (Ref. 11).

The constraint method gave the exact solution when the 5th order polynomial was employed, and only one finite element was necessary. The results obtained by the 18 degrees-of-freedom high-precision element (also based on the 5th order polynomial) for various finite element layouts were presented in reference 6 for displacements and bending moments at the centroid of the plate. Figure 1(b) shows the layout given in reference 6 for  $N=1$  and  $N=36$ , where  $N$  is the total number of elements. Finite element layouts ( $N=25$  and  $100$ ) used in the NASTRAN model are shown in figure 1(c). Due to symmetry, only one-half of the plate was considered. Results at the centroid of the plate are given in table 1. It is seen that 36 high-precision elements with 108 degrees-of-freedom (DOF) were needed to obtain precision to five significant digits whereas only 6 DOF were needed in the constraint method to achieve similar precision. The NASTRAN results were obtained by interpolation. Employing 100 CTRPLT elements with 166 DOF, 10 percent error was observed. Additional comparisons between NASTRAN plate elements and the constraint method are presented in figures 2 and 3 for the displacement and bending moment  $M_x$  along the centerline of the plate, respectively. The NASTRAN

100-element model gave satisfactory answers for the displacements but only marginal accuracy for moment. Similar accuracy was observed for the bending moment  $M_y$  along the same line.

The second example is a rectangular plate with two opposite edges simply supported, the third edge free, and the fourth edge fixed under uniform pressure  $q$  (Fig. 4(a)). This is an interesting problem because it comprises all common boundary conditions. Due to symmetry, only one-half of the plate was considered. Finite element layouts are shown in figures 4(b) and 4(c) for the constraint method and the NASTRAN model, respectively. The quadrilateral bending element QDPLT was used in the NASTRAN model with 300 DOF. Results obtained by the constraint method were also reported elsewhere (Ref. 7). Rapid convergence was observed with respect to increasing orders of approximation. It was found that very good results were obtained for the 6th order approximation with 21 DOF (free variables). These are compared with the NASTRAN results in figures 5 and 6 for bending moments along a line in the middle of the rectangular plate. It is seen that correlation of the NASTRAN results for  $M_y$  with the exact solution is not as good as for  $M_x$ . In this case the NASTRAN model overestimates the maximum  $M_y$  by about 50%. It should be noted, however, that NASTRAN gave satisfactory results along the centerline of the plate.

#### Dynamic Analysis

The first example for dynamic application is a cantilevered triangular plate. Natural frequency of the plate was solved by the constraint method for various combinations of finite element layouts and orders of approximation. Results are given in table 2 together with the results obtained by the high-precision 18 degrees-of-freedom plate element and the experimental data (Ref. 12). The results show that in the constraint method monotonic convergence can be achieved by increasing the orders of approximation as well as by reducing element sizes. It is noted that the DOF represent the total number of equations in the associated eigenvalue problems. Comparable results were obtained by the constraint method with fewer DOF.

The next dynamic problem is the free vibration of a simply-supported square plate shown in figure 7(a). Two elements were used for one-half of the plate in the constraint method (Fig. 7(b)), and 200 elements in the NASTRAN model (Fig. 7(c)). Natural frequencies of the first three modes are presented in table 3. Monotonic convergence was obtained by increasing the orders of approximation in the constraint method. The NASTRAN results, presented in reference 13, are also given for comparison. It is significant that the resulting number of DOF for the eigenvalue problem is much smaller in the constraint method.

## IMPLEMENTATION

Implementation of the constraint method in conjunction with the solution algorithm given in Appendix A may be divided into the following steps:

### 1. Define structural model

- a. Joint coordinates
- b. Element incidence (including order of approximation that can be provided by default value)
- c. Element compatibility (this data can be generated automatically from Element Incidence or by user's input)
- d. Element and material properties
- e. Applied loads (referred to individual element ID and define point of application by its coordinates or joint ID if the joint exists; only element ID is required for distributed load)
- f. External boundary condition (referred to individual element ID and define locations for point supports; define element boundary number for line support)

### 2. Generate and assemble matrix

- a. Unconstrained stiffness matrix  $[S]_K$ ,  $[S]$  (one functional routine for each element type of any order of approximation)
- b. Unconstrained mass matrix  $[M]_K$ ,  $[M]$  (one functional routine for each element type of any order of approximation)
- c. Unconstrained load vector  $\{Z\}_K$ ,  $\{Z\}$  (point load, uniform or nonuniform distributed load for any other order of approximation)
- d. Constraint matrix  $[P]$  (two parts: interelement compatibility and external boundary conditions)
- e. Enforced displacement vector  $\{R\}$  (null or constant value)

### 3. Determine the rank of $[P]$ and separate independent and dependent columns in $[P]$ into matrices $[B]$ and $[C]$ . This can be accomplished by using the product form of inverse to obtain $[B^{-1}]$ directly.

### 4. Constrained matrix generation

- a. Compute transformation matrices  $[H]$  (equation (A7)) and  $\{h\}$  (equation (A8))
- b. Constrained stiffness matrix  $[\bar{S}]$  (equation (7))
- c. Constrained mass matrix  $[\bar{M}]$  (equation (8))
- d. Constrained load vector  $\{\bar{Z}\} = [H^T] \{Z\}$
- e. Constrained enforced displacement vector  $\{\bar{R}\} = [H^T] [S] \{h\}$

5. Equation solver

- a. For static problem, solve  $[\bar{S}] \{a_c\} = \{\bar{Z}\} - \{\bar{R}\}$  for  $\{a_c\}$ . Compute  $\{a_b\}$  (equation (A5)) and then  $\{a\}$  (equation (A4)). Separate  $\{a\}$  into  $\{a_k\}$ ,  $k=1, 2, \dots, N$ , for each individual element.
- b. For dynamic problem, solve  $[\bar{S}] \{a_c\} + [\bar{M}] \{\ddot{a}_c\} = 0$

6. Output data processing

- a. Compute results for each element directly from the approximating polynomials whose coefficients are determined in step 5.
- b. Users define the element ID and desired locations of output recovery; some default values may be provided.

### NASTRAN Implementation

In executing these operations, step 1 requires some modification of NASTRAN procedures. In particular, the element compatibility data needed in constructing the constraint matrix [P] in step 2, and the options for specifying uniform line support conditions must be revised. Steps 2 through 4 are new except that the current multiple-point constraints and enforced displacement in NASTRAN can be included in steps 2d and 2e. The current equation solvers in NASTRAN may be used in step 5. New NASTRAN functional modules are also required for output data recovery in step 6, since the results are obtained directly from the approximating polynomials.

It should be noted that finite elements generated by the constraint method can be combined with existing elements in NASTRAN if it is so desired. In this case, the unknown variables consist of both coefficients in the assumed polynomials and nodal variable components. The elements can be connected together by the constraint equations.

### CONCLUDING REMARKS

The constraint method is an efficient and cost effective approach to finite element approximations. It reduces modeling time significantly because fewer elements are needed. The structural model thus may be generated faster and with fewer errors. The accuracy and computational efficiency are not sensitive to input numbering schemes, and remodeling is not required for greater accuracy. Results presented herein and those obtained in other test cases (Refs. 7 to 10) indicate that highly accurate results can be obtained by the constraint method at reduced computer costs. It is desirable, however, to solve some larger problems to provide better comparisons between this approach and the current approaches to finite element structural analysis.

Efficiency of this approach may be further improved by the development of efficient algorithms for obtaining  $[B^{-1}]$ . Such an effort is currently underway at Washington University in St. Louis.

Implementation of the constraint method into the existing general purpose computer program such as NASTRAN is considered feasible and worthy of further investigation.

APPENDIX A

A SOLUTION ALGORITHM FOR THE CONSTRAINT METHOD

The problem is to minimize the total potential energy (equation (A1)) subject to a set of constraints (equation (A2)):

$$\text{Min. } \pi = \frac{1}{2} \{a\} [S] \{a\} - [Z] \{a\} \quad (\text{A1})$$

$$\text{Subject to: } [P] \{a\} = \{R\} \quad (\text{A2})$$

We begin by selecting  $m$  linearly independent columns from  $[P]$  and renaming them  $[B]$ . Then equation (A2) can be written as .

$$[B] \{a_b\} + [C] \{a_c\} = \{R\} \quad (\text{A3})$$

where vector  $\{a_b\}$  contains the variables associated with the linearly independent columns in  $[B]$ , and  $\{a_c\}$  contains the remaining variables in  $\{a\}$ . Vector  $\{a\}$  is related to  $\{a_b\}$  and  $\{a_c\}$  by the following equation

$$\{a\} = [T] \begin{Bmatrix} a_b \\ a_c \end{Bmatrix} \quad (\text{A4})$$

in which  $[T]$  is the appropriate permutation matrix.

From equation (A3), we can write

$$\{a_b\} = [B^{-1}] \{R\} - [B^{-1}] [C] \{a_c\} \quad (\text{A5})$$

Substituting equation (A5) into equation (A4), vector  $\{a\}$  can be expressed in terms of  $\{a_c\}$  as

$$\{a\} = [H] \{a_c\} + \{h\} \quad (\text{A6})$$

where

$$[H] = [T] \begin{bmatrix} -[B^{-1}] [C] \\ [I] \end{bmatrix} \quad (\text{A7})$$

$$\{h\} = [T] \begin{Bmatrix} [B^{-1}] \{R\} \\ \{0\} \end{Bmatrix} \quad (\text{A8})$$

Substituting equation (A6) into equation (A1), the total potential energy  $\pi$  can be written as

$$\begin{aligned} \pi = & \frac{1}{2} \{a_c\} [H^T] [S] [H] \{a_c\} + \\ & \{a_c\} [H^T] [S] \{h\} + \frac{1}{2} \{h\} [S] \{h\} \\ & - \{a_c\} [H^T] \{z\} - \{h\} \{z\} \end{aligned} \quad (A9)$$

Minimizing  $\pi$  with respect to the elements of  $\{a_c\}$ , we have

$$[H^T] [S] [H] \{a_c\} + [H^T] \left( [S] \{h\} - \{z\} \right) = 0 \quad (A10)$$

Equation (A10) represents a set of simultaneous algebraic equations. It is noted that the original  $n$  variables in  $\{a\}$  were reduced to  $n-m$ , where  $m$  is the rank of the constraint matrix  $[P]$ . When the boundary displacements vanish,  $\{R\}$  is null. Equation (A10) then becomes

$$[H^T] [S] [H] \{a_c\} - [H^T] \{z\} = 0 \quad (A11)$$

Once  $\{a_c\}$  is solved,  $\{a_b\}$  can be obtained from equation (A5).

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TABLE 1 SOLUTION FOR S.S. EQUILATERAL TRIANGULAR PLATE

| METHOD                    | NO. OF ELEMENTS | DEGREES OF FREEDOM | DISPLACEMENT AT CENTROID | BENDING MOMENT AT CENTROID |
|---------------------------|-----------------|--------------------|--------------------------|----------------------------|
| CONSTRAINT METHOD (EXACT) | 1               | 6                  | 1.02880                  | 2.40740                    |
| COWPER                    | 1               | 3                  | .617284                  | 1.08333                    |
|                           | 36              | 108                | 1.02881                  | 2.40792                    |
| NASTRAN                   | 25              | 46                 | .92                      | 1.78                       |
|                           | 100             | 166                | .99                      | 2.19                       |

TABLE 2 NATURAL FREQUENCY OF CANTILEVERED TRIANGULAR PLATE (STEEL, T = .061")

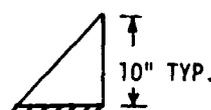
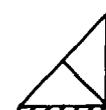
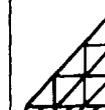
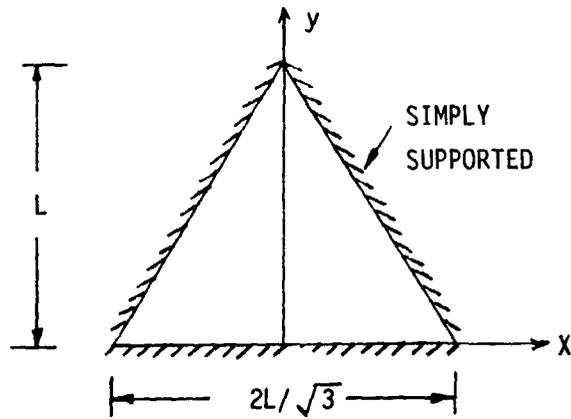
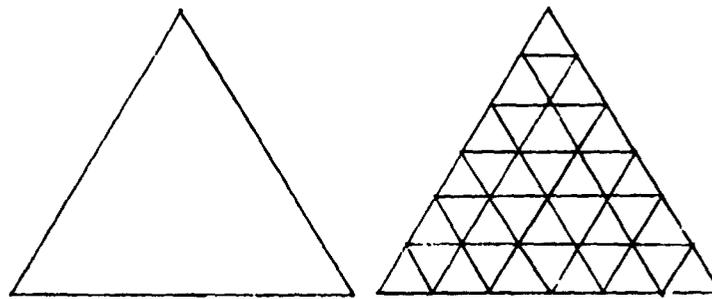
| FINITE ELEMENT LAYOUTS | CONSTRAINT METHOD  |         |         |  |          |          | COWPER (REF. 6)  |  | EXPERIMENT (REF. 12) |      |
|------------------------|--|---------|---------|--|----------|----------|--|--|----------------------|------|
|                        |  |         |         |  |          |          |  |  |                      |      |
| ORDER OF APPROX.       | 4TH  | 5TH     | 6TH     | 4TH  | 5TH      | 6TH      | 5TH  |  |                      |      |
| D O F                  | 6  | 10      | 15      | 12   | 20       | 30       | 36   | 60   |                      |      |
| MODE NO.               | 1  | 36.8757 | 36.6528 | 36.6024  | 36.5538  | 36.5331  | 36.5158  | 36.6419  | 36.6201              | 34.5 |
| 2                      | 156.983  | 144.025 | 139.187 | 141.0743   | 139.3590 | 138.9769 | 139.3265   | 139.2633   | 136                  |      |
| 3                      | 219.501  | 197.770 | 194.499 | 203.3356   | 194.0896 | 193.5854 | 194.1408   | 194.0186   | 190                  |      |

TABLE 3 NATURAL FREQUENCY OF THE SIMPLY-SUPPORTED SQUARE PLATE

| ORDER OF APPROXIMATION | CONSTRAINT METHOD |        |        | NASTRAN | EXACT  |       |
|------------------------|-------------------|--------|--------|---------|--------|-------|
|                        | 4TH               | 5TH    | 6TH    | 3RD     |        |       |
| D O F                  | 6                 | 12     | 20     | 590     |        |       |
| MODE NO.               | 1                 | .9298  | .9081  | .9069   | .9056  | .9069 |
| 2                      | 2.6972            | 2.3962 | 2.2782 | 2.2634  | 2.2672 |       |
| 3                      | 5.1170            | 4.6325 | 4.5474 | 4.5329  | 4.5345 |       |



(a) Geometry.

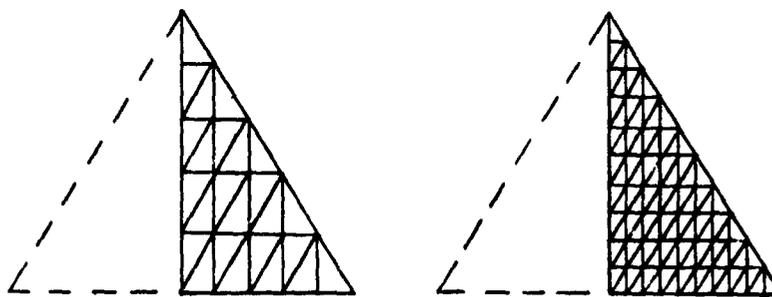


$N = 1$

$N = 36$

(b) Cowper's layouts.

NOTE:  $N =$  TOTAL NUMBER OF ELEMENTS



$N = 25$

$N = 100$

(c) NASTRAN layouts (1/2 plate).

Figure 1.- Simply-supported equilateral triangular plate.

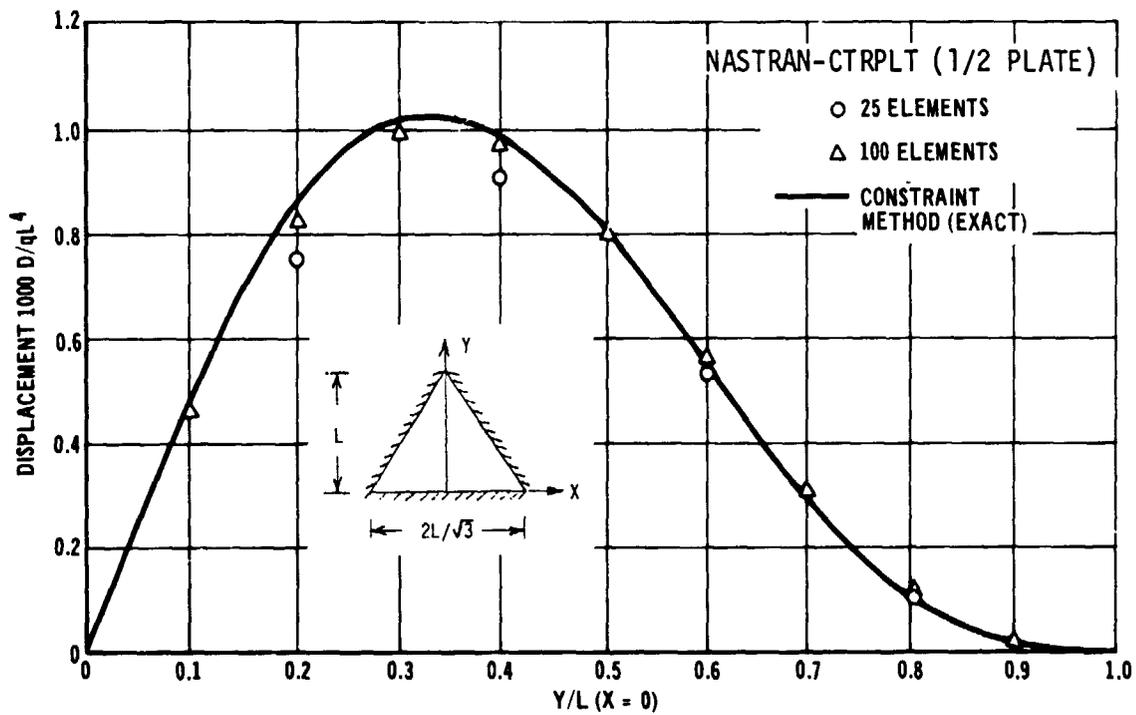


Figure 2.- Displacement along centerline of the equilateral triangular plate.

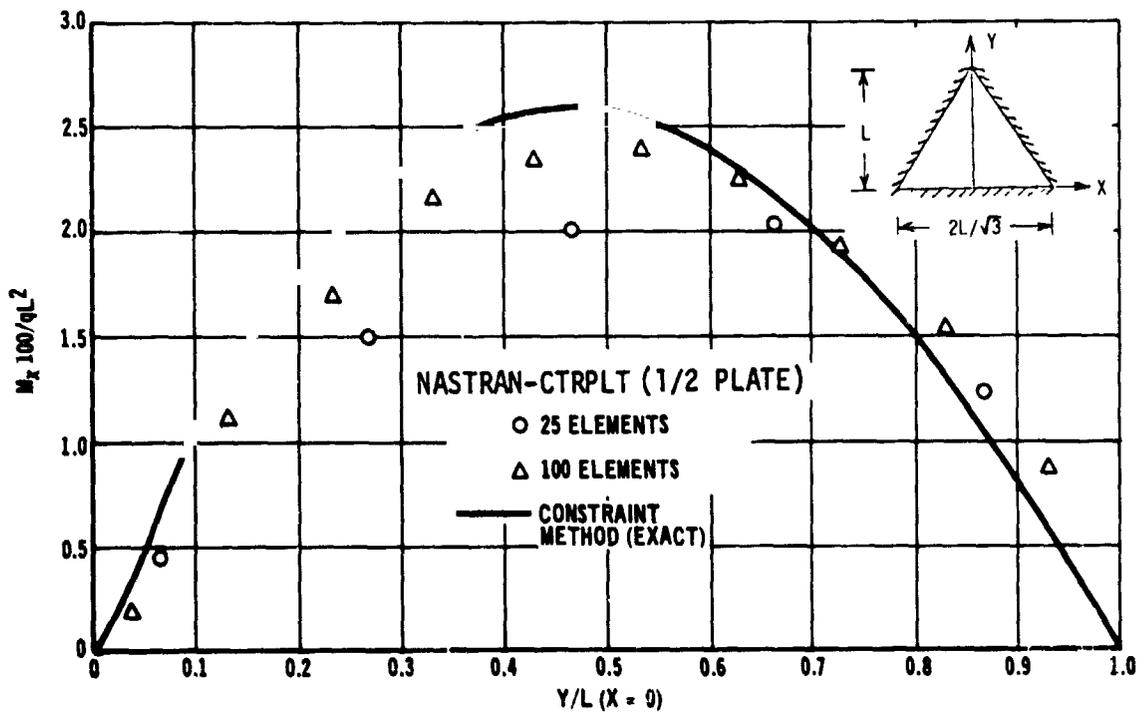
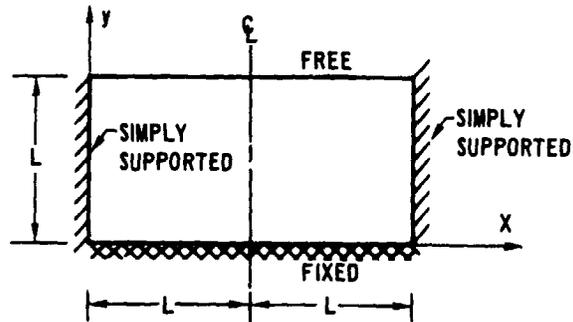
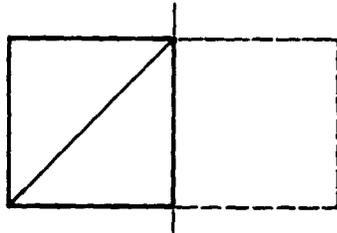


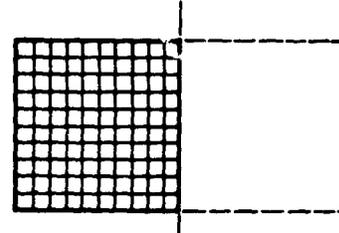
Figure 3.- Moment along centerline of the equilateral triangular plate.



(a) Geometry.



(b) Finite element layout for the constraint method.



(c) Finite element layout for NASTRAN.

Figure 4.- The rectangular plate problem.

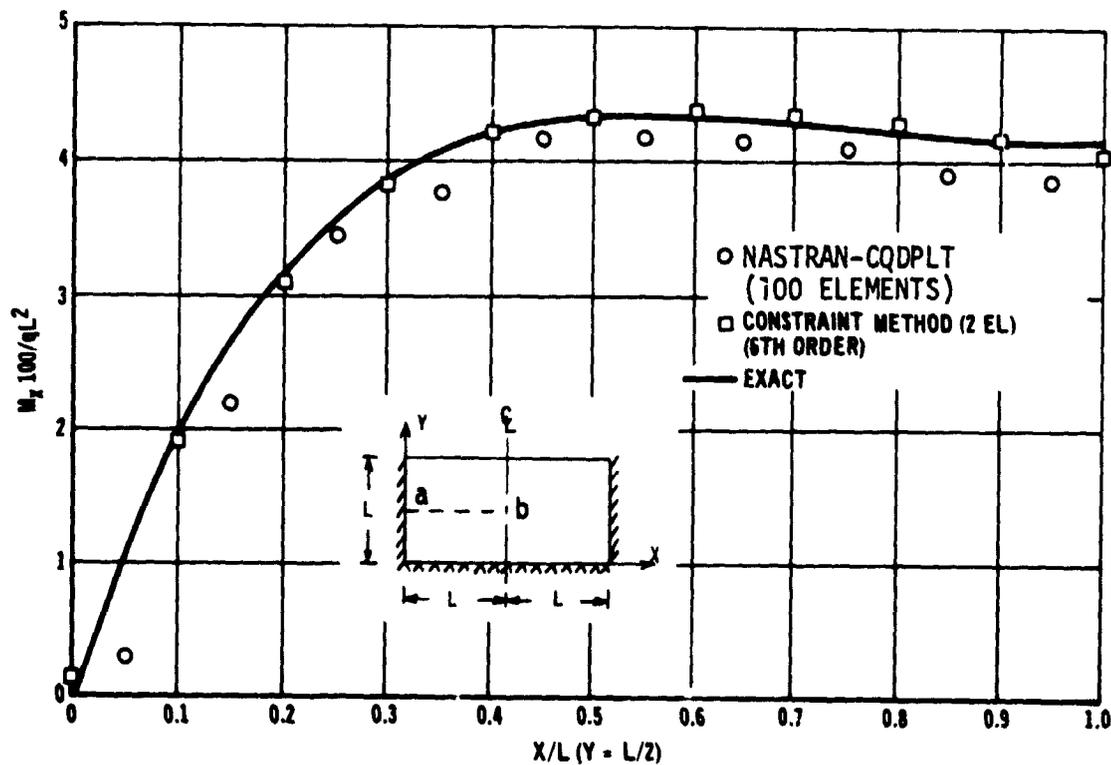


Figure 5.-  $M_x$  along line ab of the rectangular plate.

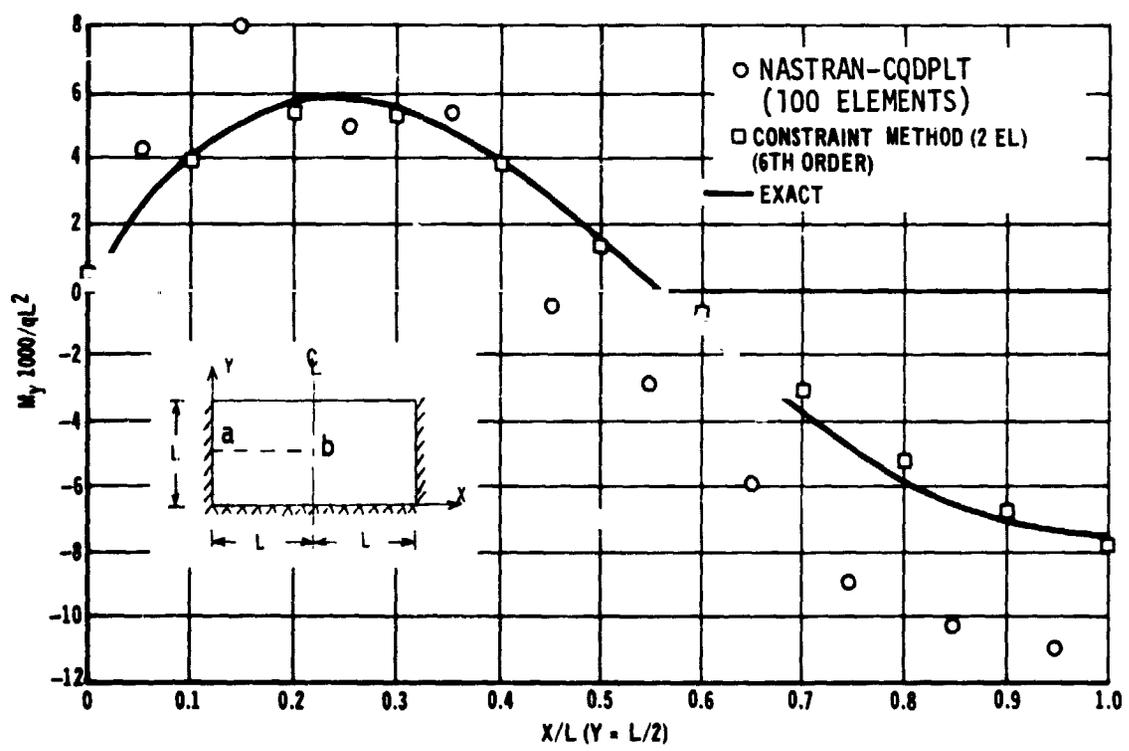
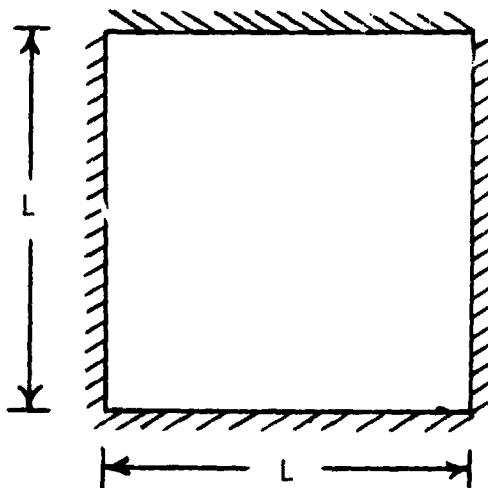
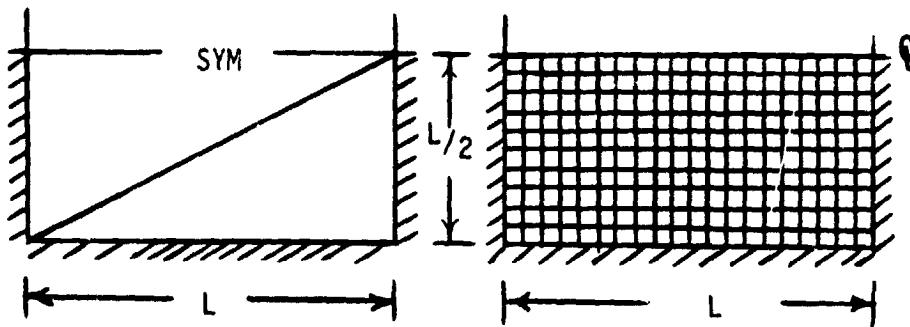


Figure 6.-  $M_y$  along line ab of the rectangular plate.



(a) Simply-supported square plate.



(b) Constraint method (2 elements). (c) NASTRAN (200 elements).

Figure 7.- Simply-supported square plate problem.